Based on K. H. Rosen: Discrete Mathematics and its Applications.

Lecture 19: Complex Counting Problems. Section 6.1

1 Complex Counting Problems

More complex counting problems cannot be solved using just the sum rule or just the product rule, but rather using a combination of both rules.

Example 1. Each user on a computer system has a password, which is six to eight characters long, where each character is an uppercase letter or a digit. Each password must contain at least one digit. How many possible passwords are there?
Ans: Let us denote by P the total number of possible passwords, and let P_6, P_7, and P_8 denote the number of possible passwords of length 6, 7, and 8, respectively. By the sum rule,

$$P = P_6 + P_7 + P_8.$$

We will now find each of the numbers P_6, P_7 and P_8. To find P_6 it is easier to find the number of strings of uppercase letters and digits that are six characters long, including those with no digits, and subtract from this the number of strings with no digits. By the product rule, the number of strings of six characters is 36^6, and the number of strings with no digits is 26^6. Hence,

$$P_6 = 36^6 - 26^6 = 2,176,782,336 - 308,915,776 = 1,867,866,560.$$

Similarly, we have

$$P_7 = 36^7 - 26^7 = 78,364,164,096 - 8,031,810,176 = 70,332,353,920$$

and

$$P_8 = 36^8 - 26^8 = 2,821,109,907,456 - 208,827,064,576 = 2,612,282,842,880.$$

Consequently,

$$P = P_6 + P_7 + P_8 = 2,684,483,063,360.$$

1.1 The subtraction rule and the division rule

The subtraction rule generalizes the sum rule:

Definition 2. (THE SUBTRACTION RULE) If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

The subtraction rule is also known as the **principle of inclusions and exclusions**. Suppose that A_1 and A_2 are sets. Then, there are $|A_1|$ ways to select an element from A_1 and $|A_2|$ ways to select an element from A_2. The number of ways to select an element from A_1 or from A_2, that is, the number of ways to select an element from their union, is the sum of the number of ways to select an element from A_1 and the number of ways to select an element from A_2, minus the number of ways to select an element in the intersection:

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

Example 3. How many natural numbers less or equal than 100 are either multiples of 5 or multiples of 7?
Ans: In general there $\lfloor 100/k \rfloor + 1$ multiples of k that are less or equal than 100. Therefore, the answer to our question is given by the inclusions and exclusions principle:

$$\lfloor 100/5 \rfloor + 1 + \lfloor 100/7 \rfloor + 1 - \lfloor 100/35 \rfloor - 1 = 21 + 15 - 3 = 33.$$

Definition 4. (THE DIVISION RULE) There are n/d ways to do a task if it can be done using a procedure that can be carried out in n ways, and for every way w, exactly d of the n ways correspond to way w.

Example 5. How many different ways are there to seat four people around a circular table, where two seatings are considered the same when each person has the same left neighbor and the same right neighbor?
Ans: We arbitrarily select a seat at the table and label it seat 1. We number the rest of the seats in numerical order, proceeding clockwise around the table. Note that are four ways to select the person for seat 1, three ways to select the person for seat 2, two ways to select the person for seat 3, and one way to select the person for seat 4. Thus, there are $4! = 24$ ways to order the given four people for these seats. However, each of the four choices for seat 1 leads to the same arrangement, as we distinguish two arrangements only when one of the people has a different immediate left or immediate right neighbor. Because there are four ways to choose the person for seat 1, by the division rule there are $24/4 = 6$ different seating arrangements of four people around the circular table.

1.2 Tree diagrams

Counting problems can be solved using tree diagrams. A tree consists of a root, a number of branches leaving the root, and possible additional branches leaving the endpoints of other branches. To use trees in counting, we use a branch to represent each possible choice. We represent the possible outcomes by the leaves, which are the endpoints of branches not having other branches starting at them. When a tree diagram is used to solve a counting problem, the number of choices of which branch to follow to reach a leaf can vary.

Example 6. The tree diagram technique can be used to find the bit strings of length four without two consecutive 1's. We see that there are eight bit strings of length four without two consecutive 1's. Can we find the number b_n of bit strings of length n with this property of no containing consecutive 1's?
For the general case, we could do some experimentation and get:

$$b_1 = 2, \quad b_2 = 3, \quad b_3 = 5 \quad \text{and in general} \quad b_{n+1} = b_n + b_{n-1}.$$

The last equality coming from the fact that a string of length $n+1$ with the property can be obtained by adding a 0 at the end of a string with length n with the property or adding 01 to a string of length $n-1$ with no consecutive 1's. Consider:
(1) How many bit strings of length n are there without two consecutive zeroes?
(2) How many bit strings of length n are there without two consecutive zeroes and without consecutive 1's?
(3) How many bit strings of length n are there without two consecutive zeroes or without two consecutive 1's?
(4) How many bit strings of length n are there with at least two consecutive zeroes and at least two consecutive 1's?